RESEARCH STATEMENT

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1. Introduction

My research interests lie in combinatorial set theory, specifically in the study of large cardinals and infinitary combinatorics. Set theory is the study of infinite objects. It began when Cantor proved that the cardinality of the reals is uncountable. This revelation led to Hilbert's first problem, the Continuum Hypothesis, asking if there is no set whose cardinality lies between the cardinality of the reals and the cardinality of the natural numbers. This problem was eventually shown to be independent of ZFC, the axioms of Zermelo-Fraenkel set theory (with Choice). Gödel showed that it is consistent for the Continuum Hypothesis to hold; Cohen introduced his method of forcing to show that it was also consistent with the axioms of ZFC for the Continuum Hypothesis to fail. These techniques kickstarted modern set theory.

Large cardinal properties arise, in part, from an attempt to generalize the properties of countable infinity to uncountable cardinals. Infinite and finite objects behave in a fundamentally different way, and large cardinals are uncountable cardinals that capture some of that behavior relative to smaller cardinals. As an example, we say a cardinal κ is *regular* if it cannot be written as the union of less than κ-many sets, each of which has size less than κ. We say that κ is a *limit cardinal* if it is has no immediate predecessor. Every successor cardinal is regular, but $\aleph_0$ is the first regular limit cardinal. Generalizing this behavior, we say that a cardinal is *inaccessible* if it is uncountable, regular, and strong limit (a generalization of limit cardinal).

Inaccessible cardinals are one of the weakest large cardinal properties, but they are strong enough that the consistency of ZFC can be proven from the existence of an inaccessible cardinal. As a consequence of Gödel's Incompleteness Theorems, we cannot prove whether or not a inaccessible cardinals exist. In fact, something stronger is true: we cannot even prove whether it is consistent with ZFC for an inaccessible cardinal to exist. Nonetheless, these large cardinals have proven to be very useful, both in set theory and in other areas - for instance, the existence of a nontrivial Grothendieck universe, a property of interest for algebraic geometers, is equivalent to the existence of an inaccessible cardinal.

Large cardinal properties have a very strong influence on the combinatorics of infinite sets. Many combinatorial properties can only hold at large cardinals, or are equiconsistent with large cardinals. I am interested in a wide variety of combinatorial properties, including the tree property and its generalizations, mutual stationarity, square principles, and stationary reflection.

My research follows two major themes. The first can be summed up by the following question: *How much large cardinal strength can consistently exist in the universe?* Research in this area consists of using Cohen's method of forcing to build models of ZFC in which combinatorial properties that follow from large cardinals hold at many successive cardinals. Of particular interest is the tree property.

König's Lemma states that every infinite tree with finite levels has an infinite branch. This leads to a natural question: does an analogous property hold for uncountable cardinals? Aronszajn showed that the theorem does not generalize to $\aleph_1$: there is an uncountable tree with countable levels that has no uncountable branch. The *tree property* is an uncountable generalization of König's Lemma. It holds at a cardinal κ if every tree of height κ with levels of size $< \kappa$ has an unbounded branch. Perhaps unsurprisingly, since König's Lemma is a statement about the relationship between the finite and the infinite, the tree property is closely linked to large cardinals: Mitchell and Silver showed that the tree property at $\aleph_2$ is equiconsistent with the existence of a kind of large cardinal called a weakly compact [17], and every weakly compact cardinal has the tree property. Since the tree property can hold at small cardinals like $\aleph_2$, despite being closely linked with large cardinals, it is a natural choice of property to focus on. A long-running program in set theory, started by Magidor, is to obtain the tree property at every cardinal simultaneously. My work in this program has focused on generalizations of the tree property that are linked with very powerful large cardinals.

The second theme is examining the relationship between different combinatorial properties. The goal is to usually to determine if two (or more) properties are incompatible, or if it is consistent with ZFC for them to coexist. If they are incompatible, this is shown by direct analysis of the properties in question. If they are compatible, this is usually shown by using forcing to build a model of ZFC in which both hold. Many combinatorial properties that are compatible nonetheless have a certain tension, and combining them requires careful forcing constructions.

In both areas, much of my research has focused on successors of singular cardinals. A cardinal κ is *singular* if it is not regular: that is, if κ can be written as the limit of fewer than κ smaller cardinals. The minimum length of such a sequence is the *cofinality* of κ. Singular cardinals famously behave in very different ways than regular cardinals. Cohen's method of violating the continuum hypothesis generalizes easily to any regular cardinal, allowing the powerset of regular cardinals to be directly controlled. On the other hand, it fails when applied to singular cardinals, and in fact controlling the powerset of singular cardinals often requires large cardinals. Some of this different behavior extends to the successors of singular cardinals. Obtaining combinatorial principles at successors of singular cardinals often requires completely different techniques and large cardinal assumptions than at other regular cardinals.

2. Generalizations of the Tree Property at Many Cardinals

One of many interesting features of the tree property is that it characterizes weakly compact cardinals up to inaccessibility. That is, an inaccessible cardinal κ (one of the weakest large cardinal properties) is weakly compact if and only if the tree property holds at κ. On the other hand, the tree property can consistently hold at small cardinals, making it a useful test case for measuring how much large cardinal strength holds at a small cardinal. A motivating question for research into the tree property is the following:

Question 2.1 (Magidor)**.** *Is it consistent for the tree property to hold at every regular cardinal greater than* $\aleph_1$*?*

There has been significant work to obtain the tree property at many successive cardinals. The program began with Mitchell [17], who obtained the tree property at $\aleph_2$; Abraham [1] forced it at $\aleph_2$ and $\aleph_3$ simultaneously, and Cummings and Foreman [6] extended this result to $\aleph_n$ for all $1 < n < \omega$. Magidor and Shelah [16] forced the tree property at $\aleph_{\omega+1}$. Finally, Neeman [18] combined these results, and showed that from countably many supercompact cardinals, it is consistent for the tree property to hold at $\aleph_n$ for all finite $n \geq 2$ and at $\aleph_{\omega+1}$ simultaneously.

My work in this area has focused on generalizations of the tree property, called the *strong tree property* and the *super tree property* (also known as ITP), that characterize stronger large cardinals in the same way. More precisely, an inaccessible cardinal κ is:

- weakly compact if and only if the tree property holds at κ,
- strongly compact if and only if the strong tree property holds at κ [13], and
- supercompact if and only if ITP holds at κ [15].

Strongly compact cardinals, and their more powerful cousins supercompact cardinals, are very strong large cardinal properties that are widely studied; generalizations of the tree property provide a valuable lens with which to examine them. Like the tree property, the strong tree property and ITP can consistently hold at small cardinals; for instance, Mitchell's forcing to obtain the tree property will also obtain the strong or super tree properties when starting from the appropriate large cardinal. If ITP holds at some small cardinal, that is strong heuristic evidence that supercompacts are in some way involved. As an example, the exact consistency strength of the Proper Forcing Axiom is a longstanding open question in set theory. Weiß [22] showed that PFA implies ITP (and a stronger principle called ISP) at $\aleph_2$; Viale and Weiß [21] used this fact to prove that all known techniques to obtain the Proper Forcing Axiom must start with a supercompact cardinal. This is some of the best evidence we have for the consistency strength of PFA.

The strong and super tree properties are much more powerful than the tree property. It is easy to violate the singular cardinal hypothesis above a cardinal with the tree property (or above a weakly compact). On the other hand, by well-known result of Solovay, SCH always holds above a strongly compact cardinal. It is open whether this also holds for cardinals with the strong or super tree property:

Question 2.2. *Suppose ITP (or the strong tree property) holds at κ. Is it consistent for the singular cardinal hypothesis to fail at a strong limit cardinal above κ?*

In light of Magidor's question, the following problem is a natural one to consider:

Problem 2.3. *Force the strong tree property or ITP at many successive regular cardinals greater than $\aleph_1$.*

Work towards this question has mirrored progress with the tree property. Fontanella [8] and Unger [20] independently obtained ITP at $\aleph_n$ for $1 < n < \omega$ simultaneously, and Hachtman and Sinapova [12] forced ITP to hold at $\aleph_{\omega+1}$.

In [2], I show that in Neeman's construction to get the tree property up to $\aleph_{\omega+1}$, ITP holds at each $\aleph_n$ and the strong tree property holds at $\aleph_{\omega+1}$. Moreover, the construction can be modified to obtain ITP at $\aleph_{\omega+1}$, at the cost of only having ITP at $\aleph_n$ for $n \geq 4$.

Theorem 2.4 (A.). *[2] Let $\langle \kappa_n \mid n < \omega \rangle$ be a sequence of supercompact cardinals. Then there is a forcing extension in which ITP holds at $\aleph_n$ for all $1 < n < \omega$, and the strong tree property holds at $\aleph_{\omega+1}$.*

Theorem 2.5 (A.). *Let $\langle \kappa_n \mid n < \omega \rangle$ be a sequence of supercompact cardinals. Then there is a forcing extension in which ITP holds at $\aleph_n$ for all $3 < n < \omega$ and at $\aleph_{\omega+1}$.*

Interestingly, there is a surprising amount of tension between ITP at $\aleph_{\omega+1}$ and at $\aleph_2$; I plan to continue investigating this, with an eye towards either generalizing Neeman's result fully to ITP, or to showing that it is impossible and ITP at $\aleph_{\omega+1}$ is incompatible with ITP at $\aleph_2$.

If the tree property and its generalizations are to hold everywhere, it must hold at successors of singular cardinals of many different cofinalities. Golshani [10] showed that the tree property can consistently hold at $\aleph_{\gamma+1}$ for any regular cardinal γ. I generalized this result [2] to the strong tree property and ITP:

Theorem 2.6 (A.). *[2] Let γ be a regular cardinal, and let $\langle \kappa_\alpha \mid \alpha < \gamma \rangle$ be a continuous increasing sequence of cardinals with $\kappa_{\alpha+2}$ supercompact for all $\alpha < \gamma$. Then there is a forcing extension in which ITP holds at $\aleph_{\gamma+1}$, and γ remains a cardinal.*

In fact, I showed that this can be done for multiple cofinalities simultaneously, albeit from a much larger large cardinal hypothesis. This had not appeared in the literature before, even for the tree property.

Theorem 2.7 (A.). *[2] Let κ_0 be supercompact. Suppose $\langle \kappa_\alpha \mid \alpha < \kappa_0 \rangle$ is a continuous increasing sequence of cardinals with $\kappa_{\alpha+2}$ supercompact for all $\alpha < \kappa_0$. Then there is a forcing extension in which ITP holds at $\aleph_{\omega_1+1}$ and the strong tree property holds at $\aleph_{\omega+1}$.*

Moreover, let $\langle \alpha_i \mid i \leq k \rangle$ be a finite sequence of regular cardinals. Then there is a forcing extension in which the strong tree property holds at $\aleph_{\omega_{\alpha_i}}$ for all $i \leq k$.

Future work in this direction seems very promising, particularly when examining the tree property and its generalizations at many successors of singulars. A natural next step is to ask if this result can be extended for infinitely many cofinalities.

Question 2.8. *Can ITP hold at infinitely many successors of singulars of different cofinality simultaneously?*

Another problem in a similar vein is to obtain ITP at all successors of singulars up to a certain point. Golshani and Hayut [11] showed that from many supercompacts it is consistent to have the tree property at arbitrary countable initial segments of successors of singular cardinals.

Question 2.9. *Can ITP hold at arbitrary countable initial segments of successors of singular cardinals?*

One might also attempt to combine the two results.

Question 2.10. *Can the tree property (or its generalizations) hold at many successors of singular cardinals of multiple cofinalities simultaneously?*

More generally, the tree property is quite well-studied, but less is known about the strong tree property and ITP. While techniques used to obtain the strong tree property or ITP are similar to those required for the tree property, the extra strength of ITP requires more complex and subtle approaches. As an example, there is a tension between ITP at $\aleph_2$ and at $\aleph_{\omega+1}$ that is completely absent for the tree property. There are a vast array of results about the tree property, and determining which results generalize to stronger properties will provide great insight both into the properties themselves, and into the large cardinals that they characterize.

Question 2.11. *What theorems about the tree property generalize to the strong tree property or ITP?*

A third avenue of research that I plan to explore examines the connection between these properties and the associated large cardinals. Strongly compact and supercompact cardinals are well-studied, and they have many interesting combinatorial properties. This leads to the following general question:

Question 2.12. *What properties of strongly compact and supercompact cardinals generalize to the strong tree property and ITP?*

3. Combining Properties at Successors of Singular Cardinals

The second major area of my research is the interplay between different combinatorial properties, particularly at successors of singular cardinals. In particular, I am interested in exploring the interplay between compactness and incompactness properties. Compactness properties are properties where the behavior of cardinals below κ determine the behavior of κ. The tree property is an example of compactness: if there are branches of height α for all $\alpha < \kappa$, then there must be a branch of height κ. Compactness properties generally follow from large cardinals. Incompactness behavior is the opposite, when a property holds below κ but fails at or above κ. Compactness and incompactness properties have a certain tension, making it difficult to combine them.

Perhaps the most prominent combinatorial property related to singular cardinals is the failure of the singular cardinal hypothesis (SCH). The SCH states that if κ is a singular strong limit cardinal and $2^\alpha < \kappa$ for all $\alpha < \kappa$, then $2^\kappa = \kappa^+$. The failure of SCH is an example of incompactness: although the cardinals below κ have relatively small powersets, κ itself has a large powerset. This incompactness is even more pronounced if GCH holds below κ.

A natural property to combine with the failure of SCH at a singular cardinal κ is the tree property and its generalizations at κ^+. This combination has been obtained for ITP (and thus the normal and strong tree properties) at $\aleph_{\omega^2}$ [7], starting from large cardinals, but it remains open whether it can be achieved at $\aleph_w$. Most forcings to obtain the failure of SCH will force combinatorial properties incompatible with the tree property, and it is not known how to reduce the constructions that doesn't down to $\aleph_\omega$.

Question 3.1 (Woodin)**.** *Is it consistent for the singular cardinal hypothesis to fail at $\aleph_w$ and the tree property to hold at $\aleph_{\omega+1}$, with $\aleph_\omega$ strong limit?*

We now turn our attention to stationary sets.

Definition 3.2. Let κ be a regular cardinal. A set is *stationary in* κ if it intersects every closed and unbounded subset of κ.

Stationary sets have another equivalent (but more technical) characterization:

Fact 3.3. *Let κ be regular and let $\kappa < \lambda$. A subset $S \subset \kappa$ is stationary iff for every algebra $\mathfrak{A}$ on λ, there is an elementary submodel $N \prec \mathfrak{A}$ such that* $\sup(N \cap \kappa) \in S$.

We use this characterization to define when a sequence of stationary sets is mutually stationary.

Definition 3.4. Let R be a set of uncountable regular cardinals, and let $\langle S_\kappa \mid \kappa \in R\rangle$ be a sequence of sets, with each S_κ stationary in κ. We say that the sequence is *mutually stationary* if for every algebra $\mathfrak{A}$ on $\sup(R)$, there is an elementary submodel $N \prec \mathfrak{A}$ such that $\sup(N \cap \kappa) \in S_\kappa$.

Informally, a sequence of stationary sets is mutually stationary if each entry in the sequence is stationary for the same reason: the same object witnesses that every set in the sequence is stationary. This property was defined by Foreman and Magidor [9] in 2001. They used it to show the nonsaturation of the nonstationary ideal on $\mathcal{P}_{\omega_1}(\lambda)$, but the property is of interest in its own right. Many results in this area have focused on obtaining mutual stationarity for sequences of a fixed cofinality.

Definition 3.5. Let $R = \langle \kappa_n \mid n < \omega\rangle$ and let $A_n \subseteq \kappa_n$ for each $n < \omega$. We say that mutual stationarity holds at $\langle A_n \mid n < \omega\rangle$ if every sequence of stationary subsets $S_n \subseteq A_n$ is mutually stationary.

Foreman and Magidor showed that mutual stationarity always holds for any $\langle \kappa_n \cap \mathrm{cof}(\omega) \mid n < \omega\rangle$. They also showed that mutual stationarity does not hold for $\langle \aleph_n \mid \cap \mathrm{cof}(\omega_1) \mid 1 < n < \omega\rangle$ in Gödel's constructible universe L. Koepke [14] showed that from a measurable, it is consistent for mutual stationarity to hold on $\langle \aleph_{2n+1} \cap \mathrm{cof}(\omega_1) \mid 1 < n < \omega\rangle$. Most recently, Ben-Neria [5] showed that from countably many supercompact cardinals, it is consistent for mutual stationarity to hold at $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid n < \omega\rangle$ for any $k < \omega$. This can be seen as a combinatorial property of $\aleph_\omega$.

My research has focused on the interplay between mutual stationary and other combinatorial properties at the successor of a singular. Using a variation of Ben-Neria's construction, Sinapova and I built a model combining the tree property at $\aleph_{\omega+1}$ with mutual stationarity for any fixed cofinality.

Theorem 3.6 (A.-Sinapova)**.** *[4] Let $\langle \kappa_n \mid n < \omega\rangle$ be an increasing sequence of supercompact cardinals. Then there is a forcing extension in which the tree property holds at $\aleph_{\omega+1}$ and mutual stationarity holds at $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid k < n < \omega\rangle$ for all $k < \omega$.*

Starting with only three supercompact cardinals, we built a model combining the failure of the singular cardinal hypothesis and mutual stationarity for any fixed cofinality.

Theorem 3.7 (A.-Sinapova)**.** *[4] From three supercompact cardinals, it is consistent that $2^{\aleph_\omega} = \aleph_{\omega+2}$, and mutual stationarity holds for $\langle \aleph_n \cap \mathrm{cof}(\aleph_k)\rangle$ for all $k < \omega$.*

This not only combines mutual stationarity with the failure of SCH, but improves the large cardinal assumption for the mutual stationarity part from countably many supercompacts. Our construction uses standard Prikry forcing with interleaved collapses, forcing to make a large cardinal singular while simultaneously forcing to make it become $\aleph_\omega$.

Another combinatorial property concerning stationary sets is stationary reflection.

Definition 3.8. We say that a stationary subset $S \subseteq \kappa$ *reflects* if there is $\alpha < \kappa$ $S \cap \alpha$ is stationary in α. *Stationary reflection* holds at κ if every stationary subset of κ reflects.

In words, stationary reflection means that every large subset of κ has some large initial segment. Stationary reflection is a compactness property: if a subset of κ has no stationary initial segment, it cannot itself be stationary.

Poveda, Rinot, and Sinapova [19] recently showed that it was consistent to combine stationary reflection and the failure of SCH at $\aleph_\omega$. Combining a modified version of their construction with the techniques we developed in [4], I was able to build a model combining stationary reflection and the failure of SCH held at $\aleph_\omega$ with mutual stationarity for fixed cofinalities.

Theorem 3.9 (A.)**.** *[3] From a sequence of supercompacts of order type $\omega + 2$, it is consistent for stationary reflection to hold at $\aleph_{\omega+1}$, SCH to fail at $\aleph_\omega$, and mutual stationarity to hold $\langle \aleph_n \cap \mathrm{cof}(\aleph_k) \mid n < \omega \rangle$ for all $k < \omega$.*

These results provide a blueprint for obtaining mutual stationarity with Prikry-type constructions. The properties we make use of in the proofs are shared by almost all forcings of this type.

Question 3.10. *What other combinatorial properties can be combined with mutual stationarity?*

Another avenue of research is to examine the situation at singular cardinals of uncountable cofinality. Obtaining combinatorial properties at these cardinals requires more complex forcing, but variations of these techniques may prove effective.

Question 3.11. *To what extent do these mutual stationarity results generalize to cardinals of uncountable cofinality?*

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